The alternating sum-of-divisors function

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We survey arithmetic and asymptotic properties of the alternating sum-of-divisors function β defined by $\beta(p^a) = p^a - p^{a-1} + p^{a-2} - \ldots + (-1)^a$ for every prime power p^a $(a \geq 1)$, and extended by multiplicativity.
The function β, as a variation of the sum-of-divisors function σ, was considered by Martin [5], Guy [3], Iannucci [4], Zhou and Zhu [7] regarding the following problem. In analogy with the perfect numbers, n is said to be imperfect if $2\beta(n) = n$. The only known imperfect numbers are 2, 12, 40, 252, 880, 10 880, 75 852, 715 816 960 and 3 074 457 344 902 430 720 (sequence A127725 in [8]).
This function occurs in the literature also in another context. Let $b(n) = \#\{k : 1 \leq k \leq n$ and $\gcd(k, n)$ is a square$\}$. Then $b(n) = \beta(n)$ $(n \geq 1)$, see, e.g., Bege [1, p. 39], Cohen [2, Cor. 4.2], Iannucci [4, p. 12], Sivaramakrishnan [6].
We also pose some open problems. One of them is concerning super-imperfect numbers n, defined by $2\beta(\beta(n)) = n$. This notion seems not to appear in the literature. The numbers 2, 4, 8, 128, 32 768 and 2 147 483 648 are super-imperfect and there are no others up to 10^7. We discuss connections to the Fermat numbers.
The corresponding concept for the σ function is the following. A number n is called superperfect if $\sigma(\sigma(n)) = 2n$. The even superperfect numbers are 2^{p-1}, where $2^p - 1$ is a Mersenne prime (sequence A019279 in [8]). No odd superperfect numbers are known.

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