On the existence of the generalized Gauss composition of means

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Let $I \subset \mathbb{R}$ be a non-void open interval. Let $M_i : I^2 \to I (i = 1, 2)$ be means on I and $a, b \in I$. Consider the sequences (a_n) and (b_n) defined by the Gauss iteration in the following way:

$$\begin{aligned} a_1 &:= a, & b_1 &:= b, \\ a_{n+1} &:= M_1(a_n, b_n), & b_{n+1} &:= M_2(a_n, b_n) \quad (n \in \mathbb{N}). \end{aligned}$$

If exist limits $\lim_{n\to\infty} a_n$, $\lim_{n\to\infty} b_n$ and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n,$$

than this common limit is called Gauss composition of the means M_1 and M_2 for the numbers a and b, and denoted by $M_1 \otimes M_2(a, b)$.

It is known, if M_1, M_2 are strict means on I, then $M_1 \otimes M_2(a, b)$ exist for every $a, b \in I$.

We give sufficient conditions for the existence of this generalized Gauss composition. We show, if M_1, M_2 (not necessarily continuous) means may be restricted by strict means, then exists they Gauss composition. Also show, that the continuousity of restrictive means is necessary. We show, that these conditions cannot be improved or changed.